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SUBDIVISIONS OF A GRAPH OF MAXIMAL DEGREE $n+1$ IN GRAPHS OF AVERAGE DEGREE $n+\epsilon$ AND LARGE GIRTH

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Dedicated to the memory of Paul Erdős

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It is proved that for every finite graph H of maximal degree $n+1\geq 3$ and every $\epsilon>0$, there is an integer $t(H,\epsilon)$ such that every finite graph of average degree at least $n+\epsilon$ and of girth at least $t(H,\epsilon)$ contains a subdivision of H.

Introduction

It was proved in [13] that every finite graph of minimum degree $\delta(G)=n+1$ and girth $\tau(G)$ large enough contains a subdivision of the complete graph on $n+2$ vertices K_{n+2}. This implies that every finite graph G of average degree $2n$ and $\tau(G)$ large enough contains a $\dot{K}_{n+2}$, where $\dot{H}$ denotes any subdivision of the graph H. This follows by the fact that every finite graph G with $\|G\|\geq n|G|>0$ contains a subgraph of minimum degree $n+1$, where $|G|$ and $\|G\|$ denote the vertex number $|V(G)|$ and the edge number $|E(G)|$ of the graph $G=(V(G),E(G))$, respectively. It was conjectured in [15] that even $\|G\|\geq\frac{n+1}{2}|G|>0$ suffices for the existence of a $\dot{K}_{n+2}$ in G, if $\tau(G)$ is large enough. The truth of this conjecture is a consequence of the following

(1.1) Theorem. *For every integer $n\geq 2$ and every $\epsilon>0$, there is a least positive integer $t(n,\epsilon)$ such that every finite graph G with $\|G\|\geq(\frac{n}{2}+\epsilon)|G|>0$ and $\tau(G)\geq t(n,\epsilon)$ contains a $\dot{K}_{n+2}$.*

Theorem (1.1) does not remain true for $\epsilon=0$, since there are n-regular graphs of arbitrarily large girth (for instance, cf. [17] Chap. 6 or [20] Chap.

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8.8). It is generalized in the following main result of this paper, where $\Delta(H)$ denotes the maximal degree of the graph H.

(1.2) Theorem. *For every finite graph H_0 with $\Delta(H_0)\geq 3$ and every $\epsilon>0$, there is a least positive integer $t(H_0,\epsilon)$ such that every finite graph G satisfying $\|G\|\geq(\frac{\Delta(H_0)-1}{2}+\epsilon)|G|>0$ and $\tau(G)\geq t(H_0,\epsilon)$ contains a subdivision of H_0.*

This result implies obviously Theorem 2 in [13], where it was shown that every finite graph G of minimal degree $\delta(G)\geq\Delta(H_0)\geq 3$ and of sufficiently large $\tau(G)$ contains an $\dot{H}_0$.

It was proved in [11], that for every finite graph H, there is an $m\in\mathbb{N}$ with the property that every finite graph G with $\|G\|\geq m|G|>0$ contains an $\dot{H}$. So we may define the real valued function $f_t(H):=inf\{r>0:$ *every finite graph G with $\|G\|\geq r|G|>0$ and $\tau(G)\geq t$ contains an $\dot{H}$*$\}$ for every finite graph H and every $t\in\mathbb{N}$. It can be shown as in [11] for $d(n)$, that for every 2-connected H, $f_t(H)$ is even the minimum of the real numbers r in the braces. For "insignificant" t we write $f(H):=f_1(H)=f_2(H)=f_3(H)$, where we should mention that "*graph*" here always means "*loopless graph without multiple edges*". Since for H with $n:=\Delta(H)\geq 3$ the existence of $(n-1)$-regular graphs of girth at least t also shows $f_t(H)\geq\frac{n-1}{2}$, we can write Theorem (1.2) in the following equivalent form.

(1.3) Theorem. *For every finite graph H_0 with $\Delta(H_0)\geq 3$, $\lim\limits_{t\longrightarrow\infty} f_t(H_0)=\frac{\Delta(H_0)-1}{2}$ holds.*

In section 3 we will determine $f_t(K_4)$ completely: We will prove there, $f_t(K_4)=\frac{\lfloor\frac{t+1}{2}\rfloor}{\lfloor\frac{t-1}{2}\rfloor}$ *for all* $t\geq 3$. For $t=3$, this had been shown by G. A. Dirac in [5] and K. Wagner in [21]. In particular, the above formula implies $f_t(K_n)=f_{t+1}(K_n)$ for $n=4$ and odd $t\geq 3$. Does this remain true perhaps for all n? (Such a result cannot be true for all graphs, since, for instance, for a circuit C of length 4, we have $f_3(C)=\frac{3}{2}$, but $f_4(C)=1$. For it is easy to show that a graph G without a $\dot{C}$ has $\|G\|\leq\frac{3}{2}(|G|-1)$, and equality occurs for graphs which arise from a triangle by successively pasting a new triangle on a vertex.)

In [14], $f(K_5)=3$ was proved. This implies $f_4(K_5)=3$, as the complete bipartite graphs $K_{3,n}$ show. No other values $f_t(K_n)$ for $n\geq 5$ are known. It was conjectured in [15] (cf. also [16]) that $f_5(K_5)=2$ holds, and from a positive answer to question (3.11) in [16], $f_{n+2}(K_{n+1})\leq\frac{n}{2}$ would follow for all $n\in\mathbb{N}$. It was proved in [2] and [9] that $f(K_n)$ grows as n^2.

For K_5^-, i.e. the complete graph K_5 minus one edge, $f(K_5^-) = \frac{5}{2}$ was proved in [18] and $f_5(K_5^-) \leq 2$ in [15]. I do not know examples which show $f_5(K_5^-) = 2$ and I would more incline to believe $f_5(K_5^-) < 2$. This latter would imply that for every integer c, there are only finitely many graphs G with $||G|| \geq 2|G| - c$ and $\tau(G) \geq 5$ which do not contain a $\dot{K}_5^-$. This statement was proved for $c \leq 5$ and conjectured for $c = 6$ in [15].

We mention some terminology and notation. Primarily, *"graph"* in this paper always means a *finite, undirected, loopless graph without multiple edges.* A *multigraph* may have multiple edges, but no loops. An edge between the vertices x and y is denoted by $[x,y]$ and for $A \subseteq V(G)$ or a subgraph $A \subseteq G, G(A)$ denotes the subgraph of G induced by A. For $A \subseteq V(G)$ and $H \subseteq G$, we write $A \cap H := A \cap V(H)$, and $x \in G$ means $x \in V(G)$. Let $G - x := G(V(G) - \{x\})$ for $x \in G$ and $G - x := (V(G), E(G) - \{x\})$ for $x \in E(G)$. A graph G is *trivial*, if $|G| = 1$. Let $V_n(G) := \{x \in G : d_G(x) = n\}, V_{>n}(G) := \{x \in G : d_G(x) > n\}$ etc., where $d_G(x)$ is the degree of x in G. The corresponding numbers are denoted by $|G|_n, |G|_{>n}$ etc.

For $A \subseteq V(G)$ or $A \subseteq G, \partial_G(A) := \{[a,b] \in E(G) : a \in A \text{ and } b \notin A\}$ and $N_G(A) := \{x : \textit{there is an } [a,x] \in \partial_G(A) \textit{ with } a \in A\}$; for $A = \{a\}$, we write $\partial_G(a)$ and $N_G(a)$. The set of components of G is denoted by $\mathcal{C}(G)$. A *branch of a tree T at $t \in T$* is the $C \in \mathcal{C}(T - e)$ not containing t for an $e \in \partial_T(t)$. For $x, y \in G$, $d_G(x,y)$ is the distance of x and y an G. Let $d(G) := \max\limits_{x,y \in G} d_G(x,y)$ denote the diameter of G and $B_m(x) := G(\{y \in G : d_G(x,y) \leq m\})$ the ball of radius m around $x \in G$. The *branch vertices* $B(\dot{G})$ of a subdivision $\dot{G}$ of G are the vertices of $\dot{G}$ which correspond to the vertices of G. H is called a *topological minor* or *topological subgraph of* G, if there is an $\dot{H} \subseteq G$. For a set S, let $\mathcal{P}_n(S) := \{S' \subseteq S : |S'| = n\}$ denote the set of all *n-subsets* of S. We delete the subscript for the graph in the notation above, if it seems clear which graph it refers to. Let $\mathbb{N}$ denote the set of positive integers and for a non-negative integer m let $\mathbb{N}_m := \{n \in \mathbb{N} : n \leq m\}$. A path P in a graph G is considered as a subgraph, but often given in the form $P : x_0, x_1 \ldots, x_l$, where $V(P) = \{x_0, x_1, \ldots, x_l\} \in \mathcal{P}_{l+1}(V(G))$ is numbered along the path P. A path $P : x_0, x_1, \ldots, x_l$ is called an *x_0, x_l-path.* An *a, X-fan F of order* n for an $a \in G$ and $X \subseteq V(G - a)$ or $X \subseteq G - a$ consists of a, x_i-paths P_i in G for $i = 1, \ldots, n$ with an n-set $X' := \{x_i : i \in \mathbb{N}_n\}$ and $V(P_i) \cap V(P_j) = \{a\}$ for $1 \leq i < j \leq n$. We call X' the *set of endvertices of* F and say, *F ends in* X'. The connectivity number of a graph G is denoted by $\kappa(G)$.

2. Preliminary results

In section 3 we will reduce Theorem (1.1) for $n=2$ to the well known result that every G with $\delta(G) \geq 3$ contains a $\dot{K}_4$. This is normally due to G.A. Dirac, but was proved by H. Hadwiger before.

Theorem HD. ([4] and [7]). *Every graph G with $\delta(G) \geq 3$ contains a $\dot{K}_4$.*

(A simple proof of this result is pointed out in [12], Bemerkung 1.)

C. Thomassen noticed in [19] that large girth of a graph has the same effect for the existence of minors as large minimum degree.

Theorem T ([19]). *For every graph H, there is an integer $t(H)$ such that every graph G with $\delta(G) \geq 3$ and $\tau(G) \geq t(H)$ has H as a minor (i.e. G has a subgraph G' contractible to H).*

The main tool in our proof of Theorem (1.1) will be a result proved independently by Larman/Mani and Jung. There we need the concept of n-linked graph. A graph G is called *n-linked*, if $|G| \geq 2n$ and for every n disjoint pairs (x_j^1, x_j^2) of vertices of G (hence $\{x_j^i : j \in \mathbb{N}_n \text{ and } i \in \mathbb{N}_2\} \in \mathcal{P}_{2n}(V(G))$), there are disjoint x_j^1, x_j^2-paths in G for $j \in \mathbb{N}_n$. It is easy to see and well known that every n-linked graph is $(2n-1)$-connected. A reverse connection is given in the Theorem of Larman/Mani and Jung.

Theorem LMJ ([10] and [8]). *For every $n \in \mathbb{N}$, there is a least integer $g(n)$, so that every $g(n)$-connected (finite!) graph is n-linked.*

The only known values of g are $g(1)=1$ and $g(2)=6$([8]). One can find easy examples which show $g(n) \geq 3n-2$. On the other side, the good upper bound $g(n) \leq 22n$ was proved by B. Bollobás and A. Thomason in [1].

Now a few lemmata used in the proof of Theorem (1.2) for $\Delta(H_0) \geq 4$ in section 4.

(2.1) Lemma. *Assume $n, d \in \mathbb{N}$ and $\epsilon > 0$ satisfy $\sum_{i=0}^{d-1} (n-1)^i > \frac{1}{2\epsilon} \geq 1$. Then every graph G with $\|G\| \geq (\frac{n}{2}+\epsilon)|G| > 0$ has the property that $V_{>n}(G)$ cannot be covered by less than $\frac{\tau(G)}{4d+1}$ balls $B_d(x)$.*

Proof. The above condition on n and d implies $n \geq 2$ and $d \geq 2$. Let us suppose that the statement is not true for G and d satisfying the conditions above with n and ϵ. Then there are $x_1, \ldots, x_k \in G$ with $k < \frac{\tau(G)}{4d+1}$ so that $V' := V_{>n}(G) \subseteq \bigcup_{i=1}^k V(B_d(x_i))$. We may suppose $B_d(x_i) \cap V' \neq \emptyset$ for all $i \in \mathbb{N}_k$. Since $\|G\| > \frac{n}{2}|G|$ implies $|G|_{>n} \geq 1$, hence $k \geq 1$, we get $4d+1 < \tau(G)$. Define $B^i := G(B_d(x_i) \cup \bigcup_{x \in V' \cap B_d(x_i)} B_d(x))$ for $i \in \mathbb{N}_k$ and $F := G(\bigcup_{i=1}^k B^i)$.

Then $B^i \subseteq B_{2d}(x_i)$, hence $d(B^i) \leq 4d$ and B^i is a tree for $i \in \mathbb{N}_k$. If $\tau(F) < \infty$, then $\tau(F) \leq k(4d+1) < \tau(G)$, since $V(F)$ is the union of k subtrees $B^i \subseteq F$ of diameter at most $4d$ (as easily seen, for instance, by induction on k). This contradiction $\tau(F) < \tau(G)$ shows that F is a forest. We choose $T \in \mathcal{C}(F)$ and denote $Y := V' \cap T = \{y_1, \ldots, y_m\}$. Let T' be the minimal subtree of T containing Y. Then

$$\sum_{i=1}^{m} d_{T'}(y_i) \leq 2m - 2 < 2|Y|, \tag{1}$$

as easily seen by induction on m, since $V_1(T') \subseteq Y \neq \emptyset$. Consider any edge $[y, z] \in E(T) - E(T')$ with $y \in Y$ and let C_z be the branch of T at y containing z. We show now

$$1 + \sum_{z' \in C_z} d_G(z') < (n + 2\epsilon)|C_z|. \tag{2}$$

Proof of (2). Since $V' \cap C_z = \emptyset$ by definition of $T', d_G(z') \leq n$ for all $z' \in C_z$. If there is a $z' \in C_z$ with $d_G(z') < n$, then $1 + \sum_{z' \in C_z} d_G(z') \leq n|C_z| < (n+2\epsilon)|C_z|$. So we may assume $V(C_z) \subseteq V_n(G)$. Since $y \in Y, V(C_z) \subseteq V_n(G)$, and T is a tree containing $B_d(y)$, we get $|C_z| \geq \sum_{i=0}^{d-1} (n-1)^i > \frac{1}{2\epsilon}$ by preassumption, which implies (2). ∎

Denote $Z := N_G(Y) - V(T') \subseteq V(T)$ and $R := V(T) - (Y \cup \bigcup_{z \in Z} V(C_z))$. Then (1) and (2) imply

$$\begin{aligned} \sum_{x \in T} d_G(x) &= \sum_{i=1}^{m} d_{T'}(y_i) + \sum_{z \in Z} (1 + \sum_{z' \in C_z} d_G(z')) + \sum_{x \in R} d_G(x) \\ &< 2m + \sum_{z \in Z} (n + 2\epsilon)|C_z| + n|R| < (n + 2\epsilon)|T|, \end{aligned} \tag{3}$$

since $n \geq 2$.

Since $d_G(x) \leq n$ for all $x \in G - V(F)$ by definition of F, by adding (3) over all $T \in \mathcal{C}(F)$, we get $2\|G\| = \sum_{x \in G} d_G(x) < (n + 2\epsilon)|F| + n|G - V(F)| \leq (n + 2\epsilon)|G|$. This contradiction to our preassumption proves lemma (2.1). ∎

(2.2) Lemma. *Let $A \neq \emptyset$ be a set of vertices in the graph H with $m := \delta(H) \geq 3$ satisfying $c := |N_H(A)| < m$. Then $|A| > (m - c)\frac{(m-1)^{\lfloor \frac{\tau(H)-1}{2} \rfloor} - 1}{m-2}$ holds.*

Proof. Consider any $a \in A \neq \emptyset$ and set $t := \lfloor \frac{\tau(H)-1}{2} \rfloor$. Then $T := B_t(a) - \{[x,y] \in E(H) : d_H(x,a) = d_H(y,a) = t\}$ is a tree. There are at least $m-c$ branches of T at a which do not contain a vertex of $N_H(A)$. Every such branch B is contained in $H(A)$ and has $|B| \geq \sum_{i=0}^{t-1} (m-1)^i = \frac{(m-1)^t - 1}{m-2}$. Hence $|A| \geq 1 + (m-c)\frac{(m-1)^t-1}{m-2}$. ∎

In the next lemma we will consider a digraph, for which we will use similar notation as for graphs. We mention only that an x,y-path in a digraph is a continuously directed path from x to y.

(2.3) Lemma. *If a graph G has the property that there is an $n \in \mathbb{N}$ such that $\|G'\| \leq n|G'|$ for every $G' \subseteq G$, then there is an orientation $\overrightarrow{G} \subseteq G$ such that for all $x \in \overrightarrow{G}$, outdegree $d^+_{\overrightarrow{G}}(x) \leq n$ holds.*

Proof. For a digraph H, define $D_H(x) := \max\{d^+_H(x) - n, 0\}$ for $x \in H$ and $D(H) := \sum_{x \in H} D_H(x)$. Consider an orientation $\overrightarrow{G}$ of G such that $D(\overrightarrow{G})$ is least possible. We have only to show that $D(\overrightarrow{G}) = 0$.

Suppose $D(\overrightarrow{G}) > 0$. Then there is an $z \in G$ with $D_{\overrightarrow{G}}(z) > 0$. Consider $X := \{x \in G :$ *there is a z,x-path in* $\overrightarrow{G}\}$. Suppose there is an $x \in X$ with $d^+_{\overrightarrow{G}}(x) < n$. Then there is a z,x-path P in $\overrightarrow{G}$. Let $\overleftarrow{G}$ arise from $\overrightarrow{G}$ by reversing the orientation of all edges of P. Since $d^+_{\overrightarrow{G}}(z) > n$, we conclude $D(\overleftarrow{G}) < D(\overrightarrow{G})$, contradicting the choice of $\overrightarrow{G}$. So we have $d^+_{\overrightarrow{G}}(x) \geq n$ for all $x \in X$. Since there is no edge in $\overrightarrow{G}$ with tail in X and head in $V(G) - X$ by definition of X, we get $\|G(X)\| = \sum_{x \in X} d^+_{\overrightarrow{G}}(x) > n|X|$, since $d^+_{\overrightarrow{G}}(z) > 0$. This contradiction proves lemma 3. ∎

The condition in lemma (2.3) implies that every non-empty $G' \subseteq G$ has a vertex x with $d_{G'}(x) \leq 2n$ Hence there is obviously an orientation $\overrightarrow{G}$ of G so that $d^+_{\overrightarrow{G}}(x) \leq 2n$ for all $x \in G$. This would also be enough for the proof of Theorem (1.2).

3. Determination of $f_t(K_4)$

In this section, we will consider in detail the case $n = 2$ of Theorem (1.1) and prove that every graph G with $\tau(G) < \infty$ and

$$\|G\| > \lfloor \frac{\tau(G)-1}{2} \rfloor^{-1} (\lfloor \frac{\tau(G)+1}{2} \rfloor |G| - \tau(G))$$

contains a $\dot{K}_4$. Furthermore, for every $t\geq 3$, there are infinitely many graphs G with $\tau(G)=t$ and $\|G\|=\frac{\lfloor\frac{t+1}{2}\rfloor}{\lfloor\frac{t-1}{2}\rfloor}|G|-\frac{t}{\lfloor\frac{t-1}{2}\rfloor}$, which do not contain a $\dot{K}_4$, and we will characterize all these graphs.

But first we will show that for $n=2=\Delta(H_0)-1$, Theorem (1.1) and Theorem (1.2) are easily reduced to Theorem HD and Theorem T, respectively. In this context, the following definition is convenient. An *l-quasiedge of a multigraph* G is a path $P:x_0,\dots,x_l$ in G of length $l\geq 1$ such that $d_G(x_i)=2$ for $i=1,\dots,l-1$, and a *quasiedge of* G is an l-quasiedge of G for some $l\geq 1$. A *maximal quasiedge of* G is a quasiedge $P:x_0,\dots,x_l$ of G which is not a proper subpath of a quasiedge of G, i.e. $d_G(x_i)\neq 2$ for $i=0,l$ or x_0 or x_l has degree 2 in G, say, $d_G(x_0)=2$ and $N_G(x_0)=\{x_i: i=1,l\}$.

Proof of Theorem (1.2) for $\Delta(H_0)=3$. Consider a positive $\epsilon<1$ and define $l_0:=\max\{l\in\mathbb{N}:\frac{l}{l-1}>1+\epsilon\}\geq 2$. Let G be a graph satisfying $\|G\|\geq(1+\epsilon)|G|>0$ and $\tau(G)>2l_0$. Choose G_0 minimal in $(\{G'\subseteq G:\|G'\|\geq(1+\epsilon)|G'|>0\},\subseteq)$. Obviously, $\delta(G_0)\geq 2$. Let $P:x_0,x_1,\dots,x_l$ be a quasiedge of G_0. If $l>l_0$, we would get $\|G_0-\{x_1,\dots,x_{l-1}\}\|=\|G_0\|-l\geq(1+\epsilon)(|G_0|-(l-1))>0$, since $\frac{l}{l-1}\leq 1+\epsilon$ by the choice of l_0. Since this contradicts the minimality of G_0, we conclude $l\leq l_0$. Consider a maximal quasiedge $P:x_0,x_1,\dots,x_l$. Assume $d_{G_0}(x_0)=2$ and $N_{G_0}(x_0)=\{x_1,x_l\}$. Then $C:=P\cup[x_0,x_l]$ is a circuit of G_0 and $\|C\|=l+1\leq l_0+1<2l_0$, which contradicts $\tau(G)>2l_0$. So we have $d_{G_0}(x_0)>2$ and $d_{G_0}(x_l)>2$, and different maximal quasiedges of G_0 are openly disjoint. Let $\overline{G}_0$ arise from G_0 by replacing every maximal quasiedge of G_0 with an edge. Since l_0 is an upper bound for the length of all quasiedges, $\tau(\overline{G}_0)\geq\frac{\tau(G)}{l_0}>2$ follows easily. Therefore, $\overline{G}_0$ is a *graph* with $\delta(\overline{G}_0)\geq 3$. So $\overline{G}_0$, hence also G, contains a $\dot{K}_4$ by Theorem HD and, if we choose $\tau(G)$ large enough, it contains a topological minor H_0 by Theorem T, since for a graph H with $\Delta(H)\leq 3, H$ is a minor of G, iff H is a topological minor of G. (Of course, one could also apply Theorem 2 from [13].) ∎

We will now describe a procedure for the construction of all multigraphs without a $\dot{K}_4$, the "series-parallel graphs". Similar (better: more or less the same) constructions are found in the literature, for instance, in [6].

We get exactly all connected multigraphs without a $\dot{K}_4$, starting from K_1 and applying one of the following steps to a multigraph G yet constructed:

(M_1) *Add a non-trivial path $P:x_0,\dots,x_l$ with $V(P)\cap V(G)=\{x_0\}$ to G;*

(M_2) *Choose a quasiedge $x_0,x_1,\dots,x_l$ of G and add an x_0,x_l-path P with $V(P)\cap V(G)=\{x_0,x_l\}$ and $E(P)\cap E(G)=\emptyset$.*

Obviously, all multigraphs constructed in this way do not contain a $\dot{K}_4$. That we get all connected multigraphs without a $\dot{K}_4$ in this way, is an immediate consequence of Theorem HD. For, let G be a connected multigraph without $\dot{K}_4$. We may assume that G is a non-trivial *graph* (by (M_2)) with $\delta(G) \geq 2$ (by (M_1)). Furthermore, we may assume $|V(C) \cap V_{\geq 3}(G)| \geq 2$ for every circuit C in G by (M_2). Then the endvertices of every maximal quasiedge of G have degree at least 3 in G. So we get a connected multigraph $\overline{G}$ on $V_{\geq 3}(G)$ considering the maximal quasiedges of G as edges of $\overline{G}$. By Theorem HD, $\overline{G}$ has multiple edges and G is got from a smaller graph by application of (M_2).

In a similar way one can show that *one gets all* 2*-connected graphs without a* $\dot{K}_4$ *starting from a circuit and applying successively* (M_2), *where the added path* P *always has length at least* 2. Then all graphs in the steps of the construction are 2-connected. If G is got from a circuit by application of (M_2) m times, then $\|G\| = |G| + m$. For integers $m \geq 0$ and $k \geq 2$, let $C_{t,m}$ *for* $t = 2k-1$ *and* $t = 2k$ *be a graph of girth* t *which we get from a circuit of length* t *by applying* (M_2) m *times, where in every step the added path has length* k. It is easily seen that $C_{2k,m}$ consists of $m+2$ openly disjoint a,b-paths of length k for certain $a \neq b$ from $V(C_{2k,m})$ and that $\|C_{2k,m}\| = \frac{k}{k-1}(|C_{2k,m}| - 2)$ holds. In opposite to $C_{2k,m}$, the graph $C_{2k-1,m}$ is not uniquely determined by m and k for $m \geq 2$, but always $\|C_{2k-1,m}\| = \frac{k}{k-1}(|C_{2k-1,m}| - 1) - 1$ holds. For all integers $m \geq 0$ and $k \geq 2$, such graphs $C_{2k-1,m}$ exist, since $\tau(C_{2k-1,m}) = 2k-1$ remains true, if, applying (M_2), we take an l-quasiedge there for $l = k-1$ or $l = k$ (which is possible). These graphs $C_{t,m}$ are the extremal graphs for our problem.

(3.1) Theorem. *For every integer* $k \geq 2$ *the following statements* (a) *and* (b) *are true.*

(a) *Every graph* G *with*
  (i) $\|G\| \geq \frac{k}{k-1}(|G| - 2)$,
  (ii) $\tau(G) \geq 2k$,
  and $|G| \geq k+2$ *contains a* $\dot{K}_4$ *or equality holds in (i) and (ii) and* $G \cong C_{2k,m}$ *for an* $m \geq 0$;

(b) *Every graph* G *with*
  (i) $\|G\| \geq \frac{k}{k-1}(|G| - 1) - 1$,
  (ii) $\tau(G) \geq 2k - 1$,
  and $|G| \geq k + 1$ *contains a* $\dot{K}_4$ *or equality holds in (i) and (ii) and* $G \cong C_{2k-1,m}$ *for an* $m \geq 0$.

Proof. Define $t_0 := 2k$ and $n_0 := k+2$ in case (a) and $t_0 := 2k-1$ and $n_0 := k+1$ in case (b). The only graphs G with $|G| = n_0 - 1$ satisfying (i) and (ii) are trees, and then equality holds in (i). Hence a forest of larger order cannot satisfy (i), since the factor $\frac{k}{k-1}$ at $|G|$ is greater than 1. So (a) and (b) hold for all G with $|G| < t_0$. Obviously, (a) and (b) are also true for graphs of order t_0.

Assume Theorem 3.1 is wrong and choose a counterexample G_0 of least order. Then $|G_0| > t_0$, as we have seen above. Furthermore, $\delta(G_0) \geq 2$ holds, since for an $v \in V(G_0)$ with $d_{G_0}(v) \leq 1, G_0 - v$ would satisfy the same conditions (a) or (b) as G_0 does, and even with strict inequality in (i). But then $\dot{K}_4 \subseteq G_0 - v$ by the choice of G_0.

Suppose there are induced subgraphs G_1 and G_2 of G_0 with $G_1 \cup G_2 = G_0, |G_1 \cap G_2| \leq 1$, and $|G_i| \geq 2$ for $i=1,2$. Since $\delta(G) \geq 2, G_i$ contains a circuit, and hence $|G_i| \geq t_0$ for $i=1,2$. Since $\dot{K}_4 \not\subseteq G_i$, we have $\|G_i\| \leq \frac{k}{k-1}(|G_i|-2)$ in case (a) and $\|G_i\| \leq \frac{k}{k-1}(|G_i|-1)-1$ in case (b) for $i=1,2$ by the choice of G_0. In both the cases, the addition of the corresponding inequalities for G_1 and G_2 gives a contradiction to (i), using $\|G_0\| = \|G_1\| + \|G_2\|$ and $|G_1| + |G_2| \leq |G_0| + 1$. This shows that G_0 is 2-connected.

Hence, by the construction for 2-connected graphs without $\dot{K}_4$ given before Theorem (3.1), G_0 arises from a 2-connected graph G_0' by application of (M_2), since G_0 is no circuit by (i) and $|G_0| > t_0$. But this means that there are distinct l_i-quasiedges P_i of G_0 with the same endvertices a and b for $i=1,2$ such that $G_0 - V(P_i - \{a,b\})$ is 2-connected. Since $\tau(G_0) \geq t_0$ and $P_1 \cup P_2$ is a circuit, we may assume $l_1 \geq k$. Set $G_0' := G_0 - V(P_1 - \{a,b\})$.

Suppose $l_1 > k$. Then there is an a,b'-subpath P of P_1 of length $k+1$. Consider $G_0'' := G_0 - V(P - \{a,b'\})$. Then G_0'' satisfies (i) with strict inequality and (ii), and $|G_0''| \geq n_0 - 1$. Hence $|G_0''| \geq n_0$ by the second sentence of this proof and $\dot{K}_4 \subseteq G_0''$ by the choice of G_0. This contradiction $\dot{K}_4 \subseteq G_0$ shows $l_1 = k$. Then G_0' satisfies the conditions in (a) and (b), respectively. Since there is no $\dot{K}_4 \subseteq G_0', G_0'$ is isomorphic to $C_{t_0,m}$ for an $m \geq 0$ by the choice of G_0. But this implies $G_0 \cong C_{t_0,m+1}$, contradicting the choice of G_0. ∎

The maximal size of a graph on v vertices without a $\dot{K}_4$ and the extremal graphs (i.e. case (b) for $k=2$ in Theorem (3.1)) were determined by G.A. Dirac in [5] and by K. Wagner in [21], respectively.

Remark. It is also possible to characterize the exceptions, if we take in Theorem (3.1) in (i) the integer part of the right side.

Let $t_0, n_0 := \lfloor \frac{t_0}{2} \rfloor + 2$, and $k := \lfloor \frac{t_0+1}{2} \rfloor \geq 2$ be as in Theorem (3.1) and define $z_{t_0}(v) := \lfloor \frac{t_0}{t_0-2}(v-2) \rfloor$ for even $t_0 \geq 4$ and $z_{t_0}(v) := \lfloor \frac{t_0+1}{t_0-1}(v-1) \rfloor - 1$ for odd $t_0 \geq 3$. As in the proof of Theorem (3.1), it is easily seen that a graph G with $\|G\| \geq z_{t_0}(|G|), \tau(G) \geq t_0$, and $n_0 - 1 \leq |G| \leq n_0 - 1 + k - 2 = t_0 - 1$ must be a tree, and then $\|G\| = z_{t_0}(|G|)$ holds.

Choose $v \geq t_0$. We assume $v-(n_0-1) \not\equiv 0$ (mod $(k-1)$), say, $v-(n_0-1)=(m+1)(k-1)+r$ with $1 \leq r \leq k-2$. Then we get a graph G not containing a $\dot{K}_4$ with $|G|=v, \|G\|=z_{t_0}(v)$, and $\tau(G) \geq t_0$ by subdividing edges of $C_{t_0,m}$ and/or adding vertices of degree 1 (i.e. by identifying exactly one vertex of K_2 with a vertex of the graph yet constructed), altogether r times. On the other side, suppose G is a graph not containing a $\dot{K}_4$ with $|G|=v, \|G\|=z_{t_0}(v)$ and $\tau(G) \geq t_0$. Then it is seen in a similar way as in the proof of Theorem (3.1) that after successive deletion of at most r vertices of degree 1 we arrive at a graph, which can be constructed from a circuit by m-fold application of (M_2). ∎

4. Proof of Theorem 1.2

Of course, we may assume $\epsilon < 1$ in Theorem 1.2. Let a graph H_0 be given and let denote $v_0 := |H_0|$ and $e_0 := \|H_0\|$. We may assume $n := \Delta(H_0)-1 \geq 3$, since Theorem 1.2 was proved for $\Delta(H_0)=3$ in section 3. An upper bound for the necessary girth $t(H_0,\epsilon)$ will be specified by some numbers depending on H_0 and ϵ which we will now define in advance.

(D_1): *$c_0 := 3(g(e_0)+v_0-1)$; hence $c_0 \geq 3(3e_0-2+v_0-1) \geq 3(3n+v_0)$.*
(D_2): *$m_0 := \max\{c_0, 2\lceil f(H_0)\rceil\}$.*
(D_3): *d_0 denotes the least integer d satisfying $(d-1)(\delta_0'-2)+2(\delta_0'-1) \geq m_0$, where $\delta_0' := \max\{\frac{n+1}{2}, 3\}$; since $m_0 \geq c_0 > 3\delta_0'$, we see $d_0 \geq 3$.*
(D_4): *$b_0 := \sum_{i=0}^{d_0}(m_0-2)^i$; obviously, $b_0 > 1+2d_0$.*
(D_5): *$t_0 := 1+\lceil f(H_0)\rceil b_0$; since $f(H_0) \geq \frac{3}{2}, t_0 > 2b_0$.*
(D_6): *h_0 denotes the least integer h satisfying $(m_0-1)^{\lfloor \frac{h-1}{2} \rfloor} \geq \frac{n+\epsilon}{\epsilon} m_0 t_0 (m_0-2)+1$; hence $h_0 \geq 5$.*
(D_7): *k_0 denotes the least integer $k \geq 6(2d_0+1)$ satisfying $\sum_{i=0}^{k-1}(n-1)^i > \frac{1}{\epsilon} \geq 1$.*
(D_8): *$\tau_0 := \max\{h_0(2d_0+1), (4k_0+1)v_0\}$.*
Of course, τ_0 is a function of H_0 and ϵ. From D_7) we get $\tau_0 > 24(2d_0+1)v_0$.

Notice that all numbers defined above are integers at least 3 with the only exception of δ_0'.

Proof of Theorem 1.2 for $\Delta(H_0) \geq 4$. Suppose there is a graph G with $\|G\| \geq (\frac{n}{2}+\epsilon)|G| > 0$ and $\tau(G) \geq 2\tau_0$ which does not contain an $\dot{H}_0$. We choose G_0' minimal in $(\{G' \subseteq G : \|G'\| \geq (\frac{n}{2}+\epsilon)|G'| > 0\}, \subseteq)$. Then, obviously,

G'_0 is connected and $\delta(G'_0) > \frac{n}{2}$. Hence $\delta(G'_0) \geq 3$ for $n \geq 4$, but for $n = 3$, this means only $\delta(G'_0) \geq 2$.

Let us assume $n = 3$ and that there is an $x \in G'_0$ with $d_{G'_0}(x) = 2$. Suppose there is a $y \in N_{G'_0}(x)$ with $d_{G'_0}(y) = 2$. But this implies $\|G'_0 - \{x,y\}\| = \|G'_0\| - 3 \geq (\frac{3}{2}+\epsilon)|G'_0| - 3 > (\frac{3}{2}+\epsilon)|G'_0 - \{x,y\}|$, a contradiction to the choice of G'_0. Hence $d_{G'_0}(y) \geq 3$ for both the $y \in N_{G'_0}(x)$. We now replace every $x \in G'_0$ of degree 2 with an edge connecting the neighbours of x. In this way, we get a *graph* G_0 satisfying $\delta(G_0) \geq 3, \tau(G_0) \geq \tau_0$, and $\|G_0\| \geq (\frac{3}{2}+\epsilon)|G_0| > 0$.

Defining $G_0 := G'_0$ for $n \geq 4$, we can state

(1) *G_0 is connected and* $\delta_0 := \delta(G_0) \geq \delta'_0 = \max\{\frac{n+1}{2}, 3\}$.

In the following we will work in G_0, which satisfies $\|G_0\| \geq (\frac{n}{2}+\epsilon)|G_0| > 0$ and $\tau(G_0) \geq \tau_0$, but does not contain an $\dot{H}_0$. All notation $d(x), N(T)$ etc. is meant in G_0.

(2) *If $d(T) \leq 24 d_0$ for a $T \subseteq G_0$, then T is an induced tree and, if additionally $d(T) \geq d_0$, then $\partial(T) \geq m_0$.*

Proof. Every $T \subseteq G_0$ with $d(T) \leq 24 d_0$ is a tree and an induced subgraph, since $\tau(G_0) > 48\, d_0 + 1$ by D_8). If $d_0 \leq d(T) \leq 24\, d_0$, then there is an induced path $P \subseteq T$ of length $l \geq d_0 \geq 3$ and $\partial(P) = \sum_{x \in P}(d(x) - d_P(x)) \geq (l-1)(\delta_0 - 2) + 2(\delta_0 - 1) \geq m_0$ by (1) and D_3). Since T is an induced tree in G_0 and $\delta_0 > 1$, we conclude $\partial(T) \geq m_0$. ∎

We will proceed in a similar way as in the proof of Theorem 1 in [13]. So we have to find a partition of $V(G_0)$ into "small" trees T with $\partial(T) \geq m_0$.

(3) *There is a partition of $V(G_0)$ into trees $T_i \subseteq G_0\, (i \in I)$ with $|T_i| \leq t_0, d(T_i) \leq 2 d_0$, and $\partial(T_i) \geq m_0$ for all $i \in I$.*

Proof. Consider $V_0 := \{x \in G_0 : d(x) \geq m_0\}$ and $\mathcal{C}_1 := \{C \in \mathcal{C}(G_0 - V_0) : d(C) \leq 2\, d_0\}, \mathcal{C}_2 := \{C \in \mathcal{C}(G_0 - V_0) : d(C) > 2\, d_0\}$. Every $C \in \mathcal{C}_1$ is a tree with $|N(C)| \geq 2$ by (2) and (1). Hence, for every $C \in \mathcal{C}_1$, we can choose vertices $x_1^C \neq x_2^C$ from $N(C) \subseteq V_0$. (In particular, $\mathcal{C}_1 \neq \emptyset$ implies $V_0 \neq \emptyset$.) Since $\tau(G_0) > 4\, d_0 + 4$, we have $|N(C_1) \cap N(C_2)| \leq 1$ for all $C_1 \neq C_2$ from $\mathcal{C}_1$. Therefore, $L := (V_0, E(G_0(V_0)) \cup \{[x_1^C, x_2^C] : C \in \mathcal{C}_1\})$ is a *graph*. For every $L' \subseteq L$, we have $\|L'\| \leq f(H_0)|L'|$, since otherwise there would be an $\dot{H}_0$ in L', hence also in G_0. Therefore, we can partition $\mathcal{C}_1$ into $\mathcal{C}_1^v (v \in V_0)$ such that $v \in N(C)$ for all $C \in \mathcal{C}_1^v$ and $|\mathcal{C}_1^v| \leq \lceil f(H_0) \rceil$ by lemma 2.3. Then $T^v := G_0(\{v\} \cup \bigcup_{C \in \mathcal{C}_1^v} V(C))$ is a tree (by (2)) with $d(T^v) \leq 4\, d_0 + 2$ for $v \in V_0$, hence $\partial(T^v) \geq d(v) \geq m_0$ by definition of V_0. Since every $C \in \mathcal{C}_1$ is a *tree* with

$d(C) \leq 2d_0$ and $\Delta(C) < m_0$, we get $|C| \leq \sum_{i=0}^{d_0} (m_0-2)^i = b_0$ (cf. D_4)). This implies $|T^v| \leq 1 + \lceil f(H_0) \rceil b_0 = t_0$ (cf. D_5)) for all $v \in V_0$.

Since every connected graph H with $d(H) > 2d_0$ can be decomposed into vertex disjoint, connected, induced subgraphs H_1 and H_2 with $d(H_i) \geq d_0$ for $i = 1,2$, we can partition every T^v with $d(T^v) > 2d_0$ and every $C \in \mathcal{C}_2$ into vertex disjoint subgraphs T with $d_0 \leq d(T) \leq 2d_0$. All these subgraphs T are trees and $\partial(T) \geq m_0$ by (2). If T is a subgraph of T^v, we have seen above $|T| \leq t_0$. If T is a subgraph of a $C \in \mathcal{C}_2$, we get $|T| \leq b_0 < t_0$ as in the preceding paragraph for $C \in \mathcal{C}_1$. ∎

Let $T_x\,(x \in X_0)$ be a partition of G_0 into trees as in (3), where we have taken as index of a tree T a vertex $x \in T$. We contract every T_x to the vertex x and get so from G_0 a *graph* H on vertex set X_0 with $\tau(H) \geq \frac{\tau(G_0)}{2d_0+1} > 24$ and $\delta(H) \geq m_0$ by (3) and D_8).

Choose A in $\{A' : \emptyset \neq A' \subseteq V(H)$ and $|N_H(A')| < m_0\}$ as small as possible. Then $|A| > m_0$, since $\delta(H) \geq m_0$ and $\tau(H) > 3$. Since $|N_H(A')| \geq m_0$ for every $\emptyset \neq A' \subsetneqq A$ by the choice of A, by a well known variation of Menger's Theorem (see, for instance, Theorem 3.3.1 in [3]), one can find pairwise disjoint edges $e_b := [b, q(b)] \in E(H)$ for $b \in B := N_H(A)$ with $q(b) \in A$, since $|A| \geq m_0 \geq |B|$. Set $Q := \{q(b) : b \in B\}$.

Let F arise from $H(A \cup B)$ by contracting the edge e_b to the vertex $q(b) \in A$ for every $b \in B$. Then $\tau(F) \geq \frac{\tau(H)}{2} > 12$, in particular, F is a graph and $\delta(F) \geq m_0$, since $|N_H(b) \cap A| \geq 2$ for $b \in B$ by the choice of A. Now we prove

(4) $\kappa(F) \geq g(e_0) + v_0$.

Proof. Assume $S \subseteq V(F)$ separates F. Then there is a $C \in \mathcal{C}(F - S)$ with $|C \cap Q| \leq \frac{|Q-S|}{2}$. Returning to H by splitting the vertices $q(b) \in F$ into b and $q(b) \in H$, one gets $|N_H(V(C))| \leq \frac{|Q-S|}{2} + |S| + |S \cap Q| = \frac{|Q|}{2} + |S| + \frac{|S \cap Q|}{2} < \frac{m_0}{2} + \frac{3}{2}|S|$. But this implies $|S| > \frac{m_0}{3}$, since $|N_H(V(C))| \geq m_0$ by the choice of A and $V(C) \subsetneqq A$. Using D_2) and D_1), (4) follows, since $|F| = |A| > m_0$. ∎

Unlike the proof of Theorem 1 in [13], we face now the problem, whether there are enough vertices of degree exceeding n in $G_0(\bigcup_{x \in A \cup B} V(T_x))$, not too close together. This can be solved using Lemma 2.1 and the following result (5), where $\tilde{X} := \bigcup_{x \in X} V(T_x)$ for $X \subseteq X_0$.

(5) $\|G_0(\tilde{A} \cup \tilde{B})\| \geq (\frac{n}{2} + \frac{\epsilon}{2})|\tilde{A} \cup \tilde{B}|$.

Proof. We need the minimality of the graph G_0' considered at the beginning of the proof of Theorem 1.2. If $e = [y_1, y_2] \in E(G_0)$ took the place of the vertex

x with $N_{G_0'}(x)=\{y_1,y_2\}$, we set $\varphi(e):=x$; otherwise, set $\varphi(e):=\emptyset$. Using this notation, we define $\overline{X}:=\tilde{X}\cup\varphi(E(G_0(\tilde{X})))$ for $X\subseteq V(H)$. (Hence, $\overline{X}=\tilde{X}$ for $n\geq 4$.) Furthermore, set $\overline{\overline{A}}:=\overline{A\cup B}-\overline{B}$.

Suppose $\|G_0(\tilde{A}\cup\tilde{B})\|<(\frac{n}{2}+\frac{\epsilon}{2})|\tilde{A}\cup\tilde{B}|$. Then also

(i) $\|G_0'(\overline{A\cup B})-E(G_0'(\overline{B}))\|<(\frac{n}{2}+\frac{\epsilon}{2})|\overline{\overline{A}}\cup\tilde{B}|$

holds, since subdividing an edge maintains the inequality. By the choice of G_0' and $A,B\neq\emptyset$, we have

(ii) $\|G_0'-\overline{\overline{A}}\|<(\frac{n}{2}+\epsilon)|G_0'-\overline{\overline{A}}|$.

Addition of (i) and (ii) gives

(iii) $\|G_0'\|=\|G_0'-\overline{\overline{A}}\|+\|G_0'(\overline{A\cup B})-E(G_0'(\overline{B}))\|<(\frac{n}{2}+\epsilon)|G_0'|+(\frac{n}{2}+\frac{\epsilon}{2})|\tilde{B}|-\frac{\epsilon}{2}|\overline{\overline{A}}|$.

But $(\frac{n}{2}+\frac{\epsilon}{2})|\tilde{B}|<(\frac{n}{2}+\frac{\epsilon}{2})m_0t_0\leq\frac{\epsilon}{2}\frac{(m_0-1)^{\lfloor\frac{h_0-1}{2}\rfloor}-1}{m_0-2}$ by (3) and D_6), since $|B|<m_0$. If we apply Lemma 2.2 to $A\subseteq V(H)$, we get $|A|>(m_0-|B|)\frac{(m_0-1)^{\lfloor\frac{h_0-1}{2}\rfloor}-1}{m_0-2}\geq\frac{(m_0-1)^{\lfloor\frac{h_0-1}{2}\rfloor}-1}{m_0-2}$, since $\delta(H)\geq m_0, |B|<m_0$, and $\tau(H)\geq\frac{\tau(G_0)}{2d_0+1}\geq h_0$ by D_8). Combining with the preceding inequality, this supplies $(\frac{n}{2}+\frac{\epsilon}{2})|\tilde{B}|<\frac{\epsilon}{2}|A|\leq\frac{\epsilon}{2}|\tilde{A}|\leq\frac{\epsilon}{2}|\overline{\overline{A}}|$. Using (iii), this implies $\|G_0'\|<(\frac{n}{2}+\epsilon)|G_0'|$, contradicting the properties of G_0'. ∎

All vertices $x\in\tilde{A}$ have degree at least 3 in $\tilde{G}_0:=G_0(\tilde{A}\cup\tilde{B})$ by (1), but there may be some vertices $x\in\tilde{B}$ with $d_{\tilde{G}_0}(x)=1$. So we delete from $\tilde{G}_0$ successively all vertices of degree 1, till we come to a graph G_0^* of minimum degree at least 2. In this way, we may have got the tree $T_x'\subseteq T_x$ for $x\in A\cup B$. Since $|N_H(b)\cap A|\geq 2$ for every $b\in B, |T_x'|\geq 1$ for $x\in B$ and $T_x'=T_x$ for $x\in A$. For $a\in Q$, say, $a\in q(b)$ for $b\in B$, we define $T_a^*:=G_0(T_a\cup T_b')$ and for $a\in A-Q, T_a^*:=T_a$. Then T_a^* is a tree for all $a\in A$ by (3), since $\tau(G_0)>4d_0+2$. For $X\subseteq A$, set $X^*:=\bigcup_{x\in X}V(T_x^*)$. Then $G_0^*=G_0(A^*)$ and, obviously,

(5*) $\|G_0^*\|\geq(\frac{n}{2}+\frac{\epsilon}{2})|A^*|$

holds by (5), and we get F from G_0^* by contraction of T_a^* to a for $a\in A$.

Therefore, by (5*), Lemma 2.1, and D_7), it is not possible to cover $V_{>n}(G_0^*)$ by less then $\frac{\tau(G_0^*)}{4k_0+1}$ balls $B_{k_0}(x)$. So, successively, we can find $x_1,\ldots,x_{v_0}$ in $V_{>n}(G_0^*)$ with $d_{G_0^*}(x_i,x_j)>k_0$ for all $1\leq i<j\leq v_0$, since $\frac{\tau(G_0^*)}{4k_0+1}\geq v_0$ by D_8). Say, $x_i\in T_{s_i}^*$ with $s_i\in A$ for $i\in\mathbb{N}_{v_0}$, and set $S:=\{s_i:i\in\mathbb{N}_{v_0}\}$. Since $d(T_a^*)\leq 4d_0+1$ by (3) and $k_0\geq 6(2d_0+1)>3(4d_0+1)+2$ by D_7), $d_{G_0^*}(x_i,x_j)\geq k_0$ implies $d_F(s_i,s_j)\geq 3$. Especially, $|S|=v_0$, and we can define $x_{s_i} := x_i$. So we have found a set $S \subseteq A$ of v_0 vertices s_i which have pairwise distance at least three in F and $x_i \in T^*_{s_i} \cap V_{>n}(G^*_0)$ for all $i \in \mathbb{N}_{v_0}$. Now the proof can be completed in the same way as for Theorem 1 in [13]. For convenience of the reader, we repeat the main idea.

Consider the *tree* $\overline{T^*_s} := G^*_0(T^*_s \cup N_s)$ with $N_s := N_{G^*_0}(T^*_s)$ for $s \in S$. Then there is an x_s, N_s-fan F_s of order $n+1$ in $\overline{T^*_s}$ for every $s \in S$, since $d_{\overline{T^*_s}}(x_s) \geq n+1$ and $\delta(G^*_0) \geq 2$. Assume, F_s ends in the $(n+1)$-set $X_s = \{x^1_s, \ldots, x^{n+1}_s\} \subseteq N_s$ and $x^i_s \in T^*_{a^i_s}$ for $i \in \mathbb{N}_{n+1}$ and set $A_s := \{a^i_s : i \in \mathbb{N}_{n+1}\}$ for $s \in S$. Then $A_s \subseteq A$ is an $(n+1)$-set for $s \in S$ and we have $(\{s\} \cup A_s) \cap (\{s'\} \cup A_{s'}) = \emptyset$ for all $s \neq s'$ from S. Since $\kappa(F - S) \geq g(e_0)$ by (4), there are e_0 disjoint paths joining any arbitrarily chosen e_0 disjoint pairs from $\bigcup_{s \in S} A_s$ in $F - S$. Then there are also e_0 disjoint paths in $G^*_0 - S^*$ joining the corresponding pairs from $\bigcup_{s \in S} X_s$. Choosing the e_0 pairs from $\bigcup_{s \in S} A_s$ in an appropriate way, the union of the latter paths and the fans F_s contains an $\dot{H}_0 \subseteq G^*_0 \subseteq G_0$ with $B(\dot{H}_0) = \{x_s : s \in S\}$. (For more details confer the proof of Theorem 1 in [13].) This contradiction completes the proof of Theorem 1.2. ∎

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